

Holographic elements for Fourier transform

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Problems of designing the holographic Fourier transform elements are described. First, two different configurations for a Fourier transform setup are considered. The converging beam Fourier transform (CB-FT) is simpler than the conventional parallel beam Fourier transform (PB-FT) setup, and it appears that it should be preferred in ordinary cases. But the advantage of the conventional configuration is to make the Fourier plane free of the spherical factor. In order to obtain an exact Fourier transform, the design of holographic lens is described with respect to its optimization. A major problem is to eliminate possibly all aberrations, especially distortion for high values of spatial frequencies, therefore we have shown the advantage of curved holographic element which could be applied to Fourier transform operation.

Keywords: Fourier transform, holographic elements, phase transfer function.

1. Introduction

Among the discussions about holographic optical elements (HOE) that have appeared, some have described the holographic lens with aspheric transfer function and another one recorded on a spherical substrate for the effects of aberrations that should be eliminated. Although these considerations provide the useful results and conditions for optical design, a major problem is to correct possibly all aberrations, especially distortion for high values of spatial frequencies. One important application of HOEs is data storage and retrieval, where it is possible to replace the refractive lenses with holographic elements that offer a number of advantages for use in optical systems, in particular the ease of read in and read out of the data, and the feature of high holographic storage density.

In the literature we can find a number of papers [1]–[10] related to HOEs properties which have proved to be useful for different applications, among others for Fourier transform. In other words: the holographic elements transform a given set of waves into another set of waves and their performance is ideal when the read out geometry and the wavelength are identical to the recording geometry and its wavelength. The results of HOEs examinations appeared in the papers where the problem of holographic

lens design was studied and the idea of an off-axis holographic optical element design for Fourier transform was considered. In Latta's paper [2] the numerical values for hologram aberrations are shown, and the study centers on four types of hologram geometries: in-line, off-axis, near-image plane, and lensless Fourier transform. Here, a close parallelism between the bending characteristics of glass lenses and the construction geometry of holograms is noted; the analogy between lenses and holograms illustrates the flexibility with which the aberrations of hologram may be modified by changes in the recording geometry. FIENUP and LEONARD [4] discussed the requirements and the performance of Fourier transform holographic lenses for matched filter optical processor. The aberrations were analysed by using a holographic ray tracing computer program. A method for analytically determining HOE phase function was developed and applied to design a Fourier transform holographic element [5]–[7]. Further, the authors [8]–[13] described an optimal Fourier transform holographic lens for examining the object with an extended range of spatial frequencies. Usually, the Fourier transform HOE was recorded with the aid of computer generated hologram (CGH), and was based on an analytic solution involving optimization by minimizing the output wave front deviations. The Kedmi–Friesem method [8] is in contrast to the development described by CEDERQUIST and FIENUP [7], because of some simplifications and approximations. The holographic lens with the quadratic phase function appeared to be better than the conventional off-axis one, and may be improved by additional correction of the deviation between the desired and the actual output wave front. At last, GOTO and TERESAWA [14] discussed further methods for the improvement of the performance of the holographic optical element for Fourier transform by the selection of an image plane location for the circle of least confusion.

In this paper a survey of problems involved in holographic elements design for Fourier transform is presented. Our considerations of the Fourier transform holographic elements have concerned certain aspects of holographic memory and concentrated on the transfer function of the Fourier holographic elements that can be achieved. In fact, the complete analysis of the holographic image requires knowledge of both the recording and reconstruction beam parameters, but in many cases the aberration tolerance of particular HOE is governed by its application. For instance, in a Fourier element, spherical aberration may play a more critical role for small spatial frequencies than astigmatism. The performance of the holographic lenses is, as usual, analysed by ray tracing that can be illustrated in a vectorial form, too.

2. Fourier transform configuration

The problem of optimal configuration for Fourier transform has been described by JOYEUX and LOWENTHAL [3], and two extreme cases with respect to aberrations have been analysed: the parallel-beam (PB) and the converging-beam Fourier transform (CB-FT) illumination systems. The principle of the parallel-beam Fourier transform (PB-FT)

setup is based on the diffraction of a plane wave by the object frequency components. After diffraction the plane waves emerging from the object frequency components at different angles are focused by the holographic lens to corresponding points in the Fourier plane forming the Fourier transform of the object amplitude transmittance [1]

$$U_F(x_F, y_F) = A \iint_{-\infty}^{\infty} t_O(x_O, y_O) \exp \left[-i \frac{k}{f} (x_O x_F + y_O y_F) \right] dx_O dy_O \quad (1)$$

where A is an amplitude of plane wave incident perpendicularly on the object plane, and (x_O, y_O) , (x_F, y_F) are the coordinates of points at the object focal plane and Fourier plane, respectively. Figure 1b illustrates the parallel-beam setup applied (usually) for Fourier transform, where the diffracted plane waves emerging from the object inserted in the front focal plane are focused by the lens to the points of the back focal plane (Fourier plane).

In Figure 1a we see an equivalent imaging lens which can be converted into a Fourier transform system. This conversion may be achieved by a number of coherent point light sources at infinity which lateral locations are chosen in such a way that each point source corresponds to a particular diffracted plane wave. In other words, the all waves from light sources at infinity penetrate the input aperture inserted in the front

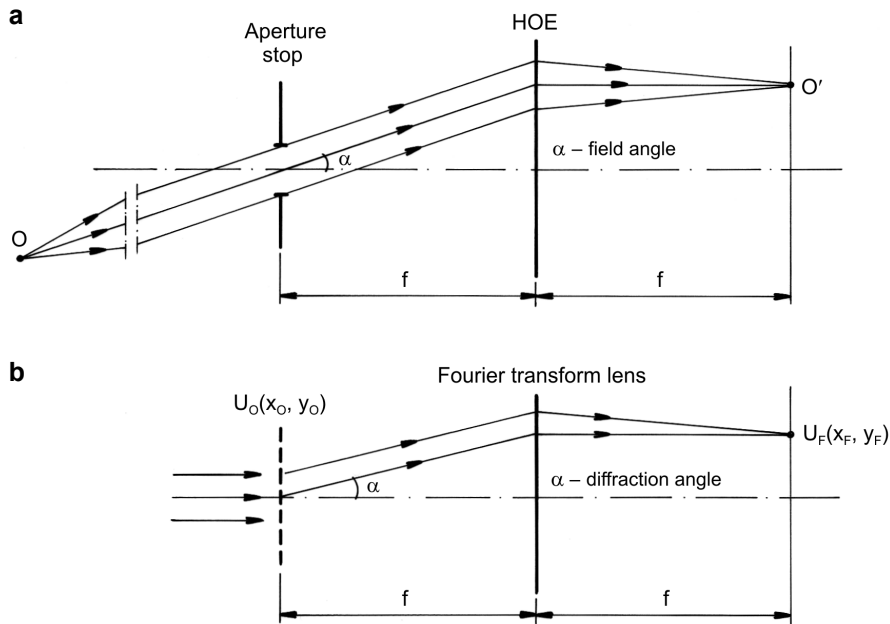


Fig. 1. Parallel-beam configuration systems: **a** – for classical imaging, **b** – for Fourier transform.

focal plane, and then are focused to different points in the back focal plane of the lens forming Fourier transform of the input amplitude transmittance. However, this lens has to be corrected for all aberrations, but is usually restricted to low aperture and rather low spatial frequencies as long as the principal rays are close enough to the optical axis. The restriction to small diffraction angles imposes severe limitations on the useful range of an optical Fourier transform setup. But the introduction of Abbe's sine condition into the principle ray path allows the paraxial restriction to be dropped, and the Fourier transform relationship becomes valid for large as well as small diffraction angles. Therefore, image formation for Fourier transform in the back focal plane of the lens must be aplanatic as well as anastigmatic, and the principle rays have to fulfill the sine condition.

The CB-FT is based on the diffraction of a converging spherical wave by the object frequency components, as shown in Fig. 2. In this case, an object is placed behind the holographic lens which is usually corrected only for spherical aberration to produce a perfect spherical wave. If the lens is illuminated by a normally incident plane wave of an amplitude A , then the incident of the object is a spherical wave converging toward the focus of the lens, and the field distribution across the focal plane is

$$U_F(x_F, y_F) = \frac{A \exp\left[i\frac{k}{2d}(x_F^2 + y_F^2)\right]}{i\lambda d} \times \iint_{-\infty}^{\infty} t_O(x_O, y_O) \exp\left[-i\frac{k}{d}(x_O x_F + y_O y_F)\right] dx_O dy_O. \quad (2)$$

Thus, up to a quadratic phase factor, the field distribution in the back focal plane is the Fourier transform of that portion of the object subtended by the projected lens aperture. The setup allows the size of spectrum to be varied, because of the spatial frequencies that are given by

$$\xi = \frac{x_F}{\lambda d}, \quad \eta = \frac{y_F}{\lambda d}. \quad (3)$$

We see that by increasing the distance d , the spatial size of the transform is increased, and by decreasing – is made smaller. Exactly, this flexibility can be of considerable utility in Fourier transform realization, especially in spatial filtering applications. The lens of CB-FT does not need to be larger than the object. Independently of the corrected spherical wave for an on-axis point, aberrations in the Fourier plane are induced by this wave owing to diffraction at the frequency components of the object. Therefore,

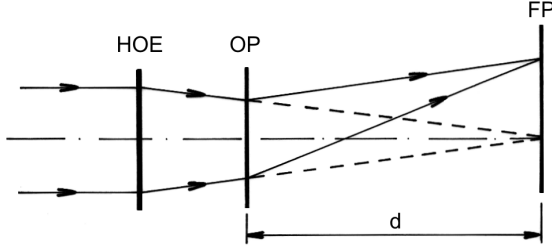


Fig. 2. Converging-beam setup for Fourier transform (HOE – holographic optical element, OP – object plane, FP – Fourier plane).

it is important to know these aberrations quantitatively. In fact, for linear holographic grating inserted in the object plane (by assuming spherical aberration $\Delta_S = 0$), the wave front deviation of third-order holographic aberrations is

$$\Delta = \Delta_C + \Delta_A + \Delta_F \quad (4)$$

where

$$\Delta_C = -\frac{\lambda \xi}{2d^2} R^3 \cos \theta,$$

$$\Delta_A = \frac{\lambda^2 \xi^2}{2d} R^2 \cos^2 \theta,$$

$$\Delta_F = \frac{\lambda^2 \xi^2}{4d} R^2$$

are the wave front deviations owing to coma, astigmatism and field curvature, respectively. R and θ are the polar coordinates in the exit pupil of the setup. The fifth monochromatic aberration, distortion, yields only a shift of image in the Fourier plane and determines the wrong position of the spot.

Analyzing aberrations of the two different configuration setups, we conclude after JOYEUX and LOWENTHAL [3] that the CB-FT should be preferred to the parallel-beam setup when their aberration levels are equivalent. It has been shown that the converging-beam configuration is a better solution for a limited range of aperture, object size and spatial frequency, and first of all it is much simpler and easier to implement. But the optical Fourier transform is generally performed by using the conventional parallel-beam configuration which is the universal setup. The advantage of this configuration is to make the Fourier plane free of the spherical factor, since in this case the phase curvature disappears leaving an exact Fourier transform relation.

3. Basic relations for optimal solution

Holographic lens recorded by plane and spherical wave fronts reconstructs an unaberrated image point in the Fourier plane only for a single spatial frequency component corresponding to that of plane recording wave front. The image point representing this spatial frequency component appears always at the proper location, but all other image points corresponding to remaining spatial frequency components are aberrated and improperly located. Therefore, for a finite set of spatial frequency components, we obtain the deviation at each of the actual output wave fronts from the desired wave front in the exit pupil of the lens. The wave front error can be computed as the mean squared wave front difference between the actual output and the desired wave fronts, and then one can determine the conditions of minimizing the wave front errors. The problem of optimizing the Fourier transform holographic lens is to find the parameters of its phase transfer function that come closest to transforming the given incident plane wave fronts into the corresponding perfect transmitted wave fronts. For any value of the diffraction angle representing a spatial frequency component of the object, there is a desired outgoing spherical wave front that converges to the corresponding image point in the Fourier plane. Therefore, if we define a set of input wave fronts $\Phi_{\text{in}}(x, y; \alpha)$ and a corresponding set of output wave fronts $\Phi_{\text{out}}(x, y; \alpha)$, then the desired phase transfer function of the holographic lens is, as follows:

$$\Phi(x, y; \alpha) = \Phi_{\text{out}}(x, y; \alpha) - \Phi_{\text{in}}(x, y; \alpha) \quad (5)$$

where α is the diffraction angle at the input transparency determining each of different members of the set of input wave fronts passing through the holographic element. But the phase transfer function of the recorded holographic lens is given by the equation

$$\Phi_H(x, y) = \Phi_O(x, y) - \Phi_R(x, y) \quad (6)$$

where $\Phi_O(x, y)$ and $\Phi_R(x, y)$ are the object and reference wave front phases, respectively at the recording medium. In practical cases it is difficult and sometimes impossible to design a perfect Fourier transform holographic lens that converts each of input plane wave fronts of the set into the corresponding ideally spherical wave front. Such a holographic lens is said to be diffraction limited, and satisfies the equation $\Phi(x, y; \alpha) = \Phi(x, y)$, where $\Phi(x, y)$ is the phase transfer function of the holographic optical element recorded by the nonspherical wave fronts and is represented by a power series of coordinates of x and y as

$$\Phi(x, y) = \frac{2\pi}{\lambda} \sum_i \sum_j a_{ij} x^i y^j. \quad (7)$$

The coefficients a_{ij} of the respective polynomials characterize the parameters of the holographic lens that can be recorded by means of the arbitrary wave fronts defined analytically or by means of computer generated holograms.

Usually, considerations comprise the wave front error that is the mean squared deviation of the actual output from the desired wave fronts at the exit pupil over the set of the respective wave fronts [6], [7],

$$E = \iint dx dy \int W(\alpha) P(x, y; \alpha) [\Phi_H(x, y) - \Phi(x, y; \alpha)]^2 d\alpha \quad (8)$$

where the pupil function $P(x, y; \alpha) = 1$ for all the points of the holographic lens, and $P(x, y; \alpha) = 0$ otherwise. Meanwhile the weighting function varies depending on the slope of different input wavefronts $0 \leq W(\alpha) \leq 1$. The goal is to find the optimal transfer phase function $\Phi_H(x, y)$ that minimizes the wave front error for all values of input parameters α specifying the spatial frequency components. Thus according to Eq. (8), the variation of the error has to satisfy the condition

$$\iint \delta \Phi_H(x, y) dx dy \int W(\alpha) P(x, y; \alpha) [\Phi_H(x, y) - \Phi(x, y; \alpha)] d\alpha = 0. \quad (9)$$

But the variation of the phase transfer function $\delta \Phi_H(x, y)$ is arbitrary, therefore

$$\int_{\alpha_1}^{\alpha_2} W(\alpha) P(x, y; \alpha) [\Phi_H(x, y) - \Phi(x, y; \alpha)] d\alpha = 0$$

for all values of x and y in the region of integration. From here we obtain the optimum phase transfer function of the holographic lens

$$\Phi_H(x, y) = \frac{\int_{\alpha_1}^{\alpha_2} W(\alpha) P(x, y; \alpha) \Phi(x, y; \alpha) d\alpha}{\int_{\alpha_1}^{\alpha_2} W(\alpha) P(x, y; \alpha) d\alpha} \quad (10)$$

which is a function of the continuous parameter α in the region, where α_1 and α_2 are its lower and upper boundary, respectively. For a special case, when a holographic lens is illuminated by a plane wave which is converted into the spherical wave focused to the point in the Fourier plane, then the phase transfer function in the meridional plane (1-D) is given by:

$$\Phi(x; \alpha) = \frac{2\pi}{\lambda} [x \sin \alpha - \sqrt{(x - f \sin \alpha)^2 + f^2}],$$

where the spatial frequency component $\xi = \sin \alpha / \lambda$ is represented by the diffracted plane wave front. Assume that $\Phi(x, \alpha)$ is continuous for $W(\alpha) = \cos \alpha$, and for the pupil function $P(x, \alpha) = 1$, then the phase transfer function (10) for the boundaries $\alpha_1 = 0$ and $\alpha_1 = \pi/6$, takes a form

$$\begin{aligned} \Phi_H(x) = & \frac{2\pi}{\lambda f} [f^2 \ln(-f\sqrt{4x^2 - 4fx + 5f^2} + 2x - f) + f^2 \ln(2f\sqrt{x^2 + f^2} - x)] \\ & + \frac{2\pi}{\lambda f} \left[\frac{2x - f}{4} \sqrt{4x^2 - 4fx + 5f^2} + \frac{x}{4}(f - 4\sqrt{x^2 + f^2}) \right]. \end{aligned} \quad (11)$$

In general, we distinguish two kinds of holographic focusing elements: i) spherical, created by means of interference of two spherical wave fronts emitted from two differently localized point sources; nota bene, the plane wave is considered also as a spherical wave of infinitely great curvature radius; ii) aspherical [5], created by means of wave fronts obtained from an auxiliary optical system (see Fig. 3) or arbitrary wave fronts determined analytically, see Eq. (7).

The holographic lenses produced by the spherical wave fronts can be used only within the region of small diffraction angles. They are often a starting point for

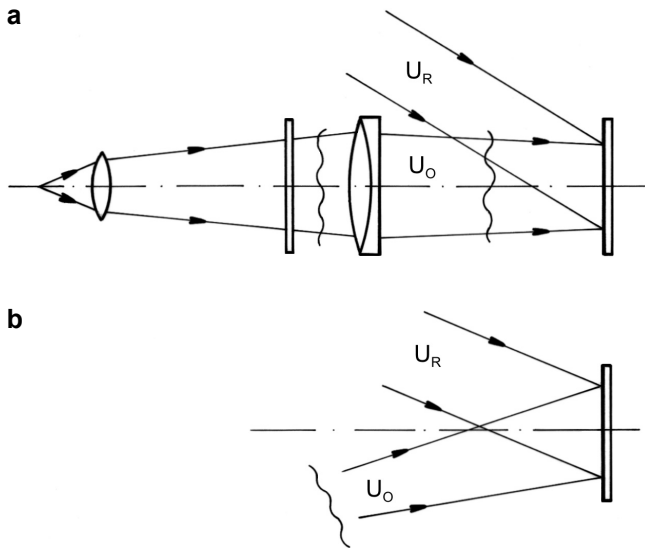


Fig. 3. Recording an aspheric holographic lens by an aspherical wave front: **a** – derived from an optical system, **b** – defined analytically.

recording the holographic optical elements to examine the spatial frequency spectrum in a broader interval. The aspheric holographic lenses can be produced by means of a plane reference wave front and a deformed spherical wave front, or with the help of a spherical wave front together with a tilted plane reference wave front to which a perturbation is introduced.

4. Holographic curved lens

Holographic optical elements are usually prepared on flat substrates, but additional degree of freedom that can reduce some of aberrations will be available by curving the element. The advantage of curving the surface of holographic element is evident, because such a lens will be aplanatic if the reference point source is at infinity and object point source is at the centre of curvature. In general, the Welford theorem [15] shows that for an arbitrary configuration of the reference and object wave fronts the HOE will be aplanatic if formed on a spherical surface with radius curvature ρ determined by the equation

$$\frac{1}{\rho} = \frac{1}{R_R} + \frac{1}{R_O} \quad (12)$$

where R_R and R_O are the distances from the hologram vertex to the reference and object point sources, respectively.

Let us consider a spherical holographic surface which vertex is in the origin of Cartesian coordinate system and its centre $C(0, 0, \rho)$ is on the z -axis, as shown in Fig. 4. The HOE has been produced by interference of the object and reference wave fronts emitting from the object point $P_O(x_O, y_O, z_O)$ and reference point source $P_R(x_R, y_R, z_R)$.

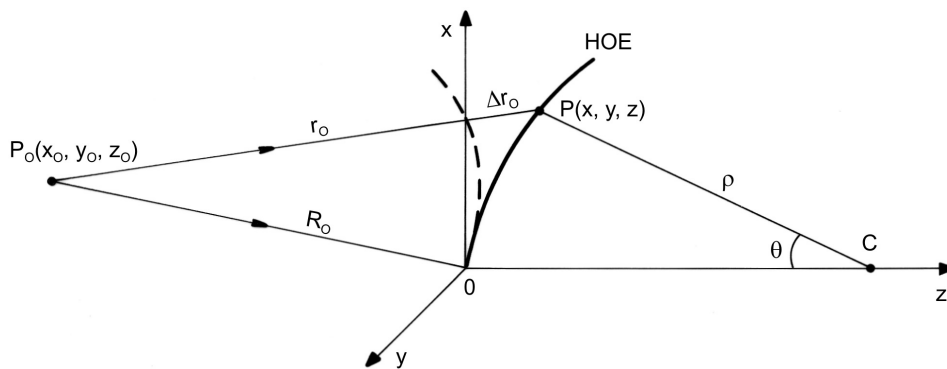


Fig. 4. Optical path difference illustrating the phase within the spherical holographic surface related to the phase at the vertex.

The points $P(x, y, z)$ at the spherical recording surface expressed in the spherical coordinates (ρ, θ, φ) , are given as:

$$x = \rho \sin \theta \cos \varphi, \quad y = \rho \sin \theta \sin \varphi, \quad z = 2\rho \sin^2\left(\frac{\theta}{2}\right)$$

where θ, φ are the angles of curvature radius with the z -axis and the radius projection on $(x-y)$ plane with x -axis, respectively. If a spherical wave front emerges from a point source, for instance from an object point $P_O(x_O, y_O, z_O)$ then its phase within the spherical surface of the recording medium related to the phase at the vertex is given in this form

$$\begin{aligned} \Phi_O(\rho, \theta, \varphi) &= \frac{2\pi}{\lambda_1}(r_O - R_O) \\ &= \frac{\pi}{\lambda_1 z_O} \left[4\rho^2 \left(1 + \frac{z_O}{\rho}\right) \sin^2\left(\frac{\theta}{2}\right) - 2\rho(x_O \cos \varphi + y_O \sin \varphi) \right] \\ &\quad - \frac{\pi}{4\lambda_1 z_O^3} \left[16\rho^4 \left(1 + \frac{z_O}{\rho}\right)^2 \sin^4\left(\frac{\theta}{2}\right) \right. \\ &\quad \left. - 16\rho^3 \left(1 + \frac{z_O}{\rho}\right) \sin^2\left(\frac{\theta}{2}\right) \sin \theta (x_O \cos \varphi + y_O \sin \theta) \right. \\ &\quad \left. + 4\rho^2 (x_O^2 \cos^2 \varphi + y_O^2 \sin^2 \varphi + 2x_O y_O \sin \varphi \cos \varphi) \sin^2 \theta \right. \\ &\quad \left. + 8\rho^2 \left(1 + \frac{z_O}{\rho}\right) (x_O^2 + y_O^2) \sin^2\left(\frac{\theta}{2}\right) \right. \\ &\quad \left. - 4\rho(x_O^2 + y_O^2)(x_O \cos \varphi + y_O \sin \varphi) \sin \theta \right] \end{aligned}$$

Analogously, the phases of reference and reconstruction wave fronts are defined as follows:

$$\Phi_R(\rho, \theta, \varphi) = \frac{2\pi}{\lambda_1}(r_R - R_R),$$

$$\Phi_C(\rho, \theta, \varphi) = \frac{2\pi}{\lambda_2}(r_C - R_C)$$

where λ_2 is the wavelength of the reconstruction wave. In fact, the reconstruction process performed by illumination of the holographic element with a spherical wave front can be described as a sum of phases

$$\frac{2\pi}{\lambda_2}(r_I - R_I) + \Phi_{ab} = \frac{2\pi}{\lambda_2}(r_C - R_C) \pm \frac{2\pi}{\lambda_1}[(r_O - R_O) - (r_R - R_R)] \quad (13)$$

where $\Phi_I(\rho, \theta, \varphi) = 2\pi(r_I - R_I)/\lambda_2$ determines the phase of the creating image wave front related to the phase at the vertex of the holographic spherical surface, and Φ_{ab} represents the wave front aberrations which are defined as the phase difference between the Gaussian reference sphere and the actual wave front created during illumination of the holographic lens. The first approximation of the Eq. (13) leads to the paraxial imaging equations, because of $\Phi_{ab}^{(1)} = 0$. Hence:

$$\frac{1}{z_I} = \frac{1}{z_C} \pm \mu \left(\frac{1}{z_O} - \frac{1}{z_R} \right),$$

$$\frac{x_I}{z_I} = \frac{x_C}{z_C} \pm \mu \left(\frac{x_O}{z_O} - \frac{x_R}{z_R} \right),$$

$$\frac{y_I}{z_I} = \frac{y_C}{z_C} \pm \mu \left(\frac{y_O}{z_O} - \frac{y_R}{z_R} \right)$$

where $\mu = \lambda_2/\lambda_1$. From these equations we see that the focusing properties of a spherical holographic lens do not depend upon the curvature of its substrate. Applying the subsequent approximation of Eq. (13), we obtain the third order aberrations described by the phase difference of the image wave front

$$\Phi_{ab}^{(3)} = \frac{2\pi}{\lambda_2} \{ (r_C^{(3)} - R_C) - (r_I^{(3)} - R_I) \pm \mu [(r_O^{(3)} - R_O) - (r_R^{(3)} - R_R)] \}. \quad (14)$$

Analyzing this equation, we see that only the spherical aberration, coma and field curvature depend on the holographic surface curvature. The radius of the HOE curvature does not influence changes of astigmatism and distortion.

The task of recording an aspheric phase transfer function on a curved substrate is rather difficult, but in some applications we can use a simple spherical wave front instead of aspheric one. If a spherical wave front tracing to an axial object point interferes with an axial plane wave front at the spherical recording medium (Fig. 5a), then the phase transfer function of such a holographic lens for Fourier transform is given as

$$\Phi_H(x, y) = \frac{2\pi}{\lambda} \left\{ [f^2 + 2(\rho - f)(\rho - \sqrt{\rho^2 - x^2 - y^2})]^{1/2} - (\rho - \sqrt{\rho^2 - x^2 - y^2}) \right\} \quad (15)$$

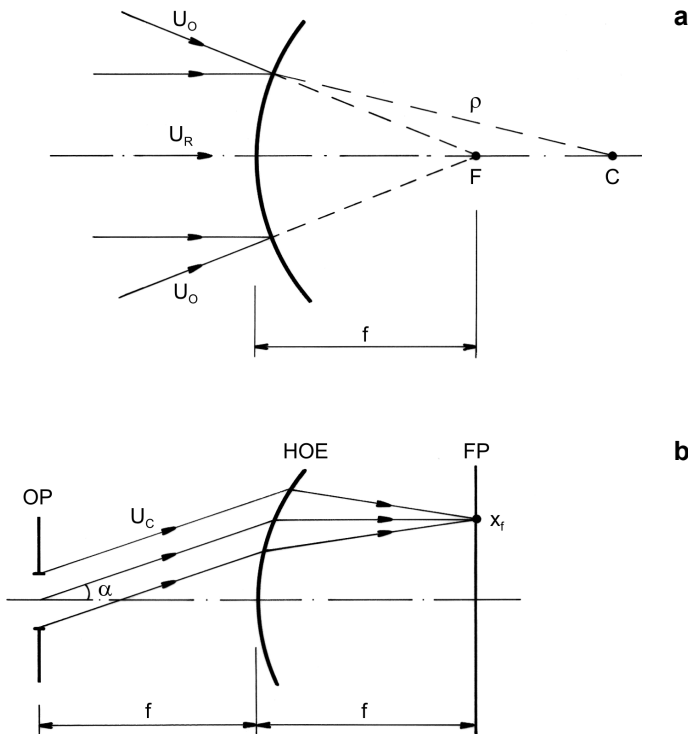


Fig. 5. Recording geometry of spherical and plane wave fronts on a spherical substrate (a), ray-tracing through a curved Fourier transform holographic lens (b).

where f is the distance of the object (focal) point from the vertex of the spherical substrate. If the focal length of this lens is equal to the curvature radius of the holographic element substrate, then the sine condition can be satisfied. In this case, the centre of the spherical holographic lens is brought into agreement with the focal point, and in contradistinction to the flat holographic lens, coma aberration is corrected (Fig. 5b).

5. Conclusions

We have presented a survey of problems involved in holographic element designing for Fourier transform. Two different configurations of a setup were considered, but an advantage of the conventional configuration is to obtain the exact Fourier transform. Thus we were able to calculate an optimal solution of phase transfer function for such a holographic lens. An interesting aspect is relatively simple holographic focusing element produced on a spherical substrate and operating in the telecentric ray tracing. It has been shown that in this case additional advantages can be achieved including the possibility of satisfying the sine condition and minimizing the coma aberration.

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